

CENTRAL COHOMOLOGY OPERATIONS AND  $K$ -THEORY

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**ABSTRACT.** For stable degree zero operations, and also for additive unstable operations of bidegree  $(0, 0)$ , it is known that the centre of the ring of operations for complex cobordism is isomorphic to the corresponding ring of connective complex  $K$ -theory operations. Similarly, the centre of the ring of  $BP$  operations is the corresponding ring for the Adams summand of  $p$ -local connective complex  $K$ -theory. Here we show that, in the additive unstable context, this result holds with  $BP$  replaced by  $BP\langle n \rangle$  for any  $n$ . Thus, for all chromatic heights, the only central operations are those coming from  $K$ -theory.

## 1. INTRODUCTION

We study cohomology operations for various cohomology theories related to complex cobordism and show that, in a suitable context, the central cohomology operations are precisely those coming from complex  $K$ -theory. Specifically, we consider the ring of additive unstable bidegree  $(0, 0)$  operations for the Adams summand of  $p$ -local complex  $K$ -theory and we show that this ring maps via an injective ring homomorphism to the corresponding ring of operations for the theory  $BP\langle n \rangle$ , for all  $n \geq 1$ . The image of this map is the centre of the target ring.

Previously results of this type had been established with target  $BP$  (which may be regarded as the  $n = \infty$  case) in both the stable and additive unstable contexts; see [3] and [6].

The  $BP\langle n \rangle$  result that we give here is quite a simple consequence of combining certain unstable  $BP$  splittings due to Wilson [8] with the results of [6]. Nonetheless we think it is interesting since it shows that the central operations are precisely those arising from  $K$ -theory at every chromatic height.

Let  $p$  be an odd prime and let  $BP$  be the  $p$ -local Brown-Peterson spectrum, a summand of the  $p$ -local complex bordism spectrum  $MU_{(p)}$ . For each  $n \geq 0$ , there is a connective commutative ring spectrum  $BP\langle n \rangle$  with coefficient groups

$$BP\langle n \rangle_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n] = BP_*/(v_{n+1}, v_{n+2}, \dots) = BP_*/J_n.$$

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Here  $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ , where the  $v_i$ s are Hazewinkel's generators, with  $v_i$  in degree  $2(p^i - 1)$  and  $J_n = (v_{n+1}, v_{n+2}, \dots)$ . These theories were introduced by Wilson in [8] and further studied by Johnson and Wilson in [4]. They fit into a tower of  $BP$ -module spectra:

$$BP \longrightarrow \dots \longrightarrow BP\langle n \rangle \longrightarrow BP\langle n-1 \rangle \longrightarrow \dots$$

In particular,  $BP\langle 0 \rangle = H\mathbb{Z}_{(p)}$  and  $BP\langle 1 \rangle = g$ , the Adams summand of connective  $p$ -local complex  $K$ -theory.

Recall that for a cohomology theory  $E$ , the bidegree  $(0, 0)$  unstable operations are given by  $E^0(\underline{E}_0)$ , where  $\underline{E}_0$  denotes the 0-th space of the  $\Omega$ -spectrum representing the cohomology theory  $E$ . Inside here we have  $PE^0(\underline{E}_0)$ , the additive bidegree  $(0, 0)$  unstable operations, which we will denote by  $\mathcal{A}(E)$ . This is a ring, with multiplication given by composition of operations.

Using unstable  $BP$  splittings due to Wilson, we will define an injective ring homomorphism  $\hat{\iota}_n : \mathcal{A}(g) \rightarrow \mathcal{A}(BP\langle n \rangle)$ . Our main result, Theorem 5.3, is that the image of  $\hat{\iota}_n$  is the centre of  $\mathcal{A}(BP\langle n \rangle)$ .

The situation is analogous to that of matrix rings, where the diagonal matrices form the centre of the  $n \times n$  matrices for all  $n$ . Indeed, we will see that all operations considered are determined by the matrices giving their actions on homotopy groups. Of course, not all matrices arise as actions of operations; there are complicated constraints. Essentially, what we show is that,

- (1) at every height  $n$ , enough matrices arise so that central operations are forced to act diagonally (in a suitable sense), and
- (2) the constraints on the diagonal operations which can occur are the same for all  $n$ .

This paper is organized as follows. In Section 2 we explain some of Wilson's results on unstable  $BP$  splittings and deduce faithfulness of the actions of additive  $BP\langle n \rangle$  operations of bidegree  $(0, 0)$  on homotopy groups. In the next section we recall some results on additive operations for the Adams summand  $g$  of connective  $p$ -local complex  $K$ -theory. We also define our map of operations  $\mathcal{A}(g) \rightarrow \mathcal{A}(BP\langle n \rangle)$  and give its basic properties. In Section 4 we define and study diagonal operations. Section 5 contains the proof of our main result, Theorem 5.3, that the image of the map coincides with the centre of the target ring.

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## 2. UNSTABLE SPLITTINGS

In this section we begin by recalling some results on unstable  $BP$  splittings. These results are due to Wilson [8]; we use [2] as our main reference. We then deduce some straightforward consequences for operations.

As usual, let  $\underline{E}_k$  denote the  $k$ -th space of the  $\Omega$ -spectrum representing the cohomology theory  $E$ . For  $n \geq 0$ , write  $\pi_n : \underline{BP}_0 \rightarrow \underline{BP}\langle n \rangle_0$  for the map coming from the map of  $BP$ -module spectra  $BP \rightarrow \underline{BP}\langle n \rangle$ . The induced map on homotopy,  $(\pi_n)_* : BP_* \rightarrow BP\langle n \rangle_* = BP_*/J_n$ , is the canonical projection.

The following lemma is the special case of [2, Lemma 22.1] for zero spaces.

**Lemma 2.1.** [2, Lemma 22.1] *For all  $n \geq 0$ , there is an  $H$ -space splitting  $\theta_n : \underline{BP}\langle n \rangle_0 \rightarrow \underline{BP}_0$  of  $\pi_n$ . Let  $e_n = \theta_n \pi_n$  denote the corresponding additive idempotent  $BP$ -operation; the choices can be made compatibly so that  $e_n e_m = e_m e_n = e_m$  for  $m < n$ .  $\square$*

These splittings immediately allow us to compare operations.

**Lemma 2.2.** *We have maps*

$$i_n : \mathcal{A}(BP\langle n \rangle) \rightleftarrows \mathcal{A}(BP) : p_n$$

*such that*

- (1)  $i_n p_n : \mathcal{A}(BP) \rightarrow \mathcal{A}(BP)$  is given by  $[f] \mapsto [e_n f e_n]$ ;
- (2)  $p_n$  splits  $i_n$  (so  $i_n$  is injective and  $p_n$  is surjective);
- (3)  $i_n$  is a non-unital ring homomorphism;
- (4)  $p_n$  is an additive group homomorphism.

*Proof.* We have the maps

$$\begin{aligned} [\theta_n \circ - \circ \pi_n] : BP\langle n \rangle^0(\underline{BP}\langle n \rangle_0) &\rightarrow BP^0(\underline{BP}_0) \\ [f] &\mapsto [\theta_n f \pi_n] \end{aligned}$$

and

$$\begin{aligned} [\pi_n \circ - \circ \theta_n] : BP^0(\underline{BP}_0) &\rightarrow BP\langle n \rangle^0(\underline{BP}\langle n \rangle_0) \\ [f] &\mapsto [\pi_n f \theta_n]. \end{aligned}$$

Since  $\pi_n$  and  $\theta_n$  are  $H$ -space maps, these maps restrict to maps on the additive operations, which we denote by  $i_n$  and  $p_n$  respectively.

The first property follows from  $\theta_n \pi_n = e_n$ . The remaining properties are easy to check using that  $\pi_n \theta_n \simeq id$  and that  $\theta_n$  is a map of  $H$ -spaces.  $\square$

**Remark 2.3.** It follows that we may  $\mathcal{A}(BP\langle n \rangle)$  identify with the subring  $e_n \mathcal{A}(BP) e_n$  of  $\mathcal{A}(BP)$ .

In the following lemma, we will record some information about actions in homotopy. Note that we can regard  $BP\langle n \rangle_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n]$  as both a subring and a quotient ring of  $BP_*$ . We will abuse notation by writing the inclusion silently and we use  $[ ]$  to denote classes in  $BP\langle n \rangle_* = BP_*/J_n$ .

- Lemma 2.4.** (1) For  $x \in BP\langle n \rangle_*$ ,  $(\theta_n)_*(x) \equiv x \pmod{J_n}$ . In particular,  $(\theta_n)_*$  is the identity in degrees less than  $2(p^{n+1} - 1)$ .
- (2) Let  $\phi \in \mathcal{A}(BP\langle n \rangle)$ . Then
- (a) for  $y \in BP_*$ ,  $(i_n\phi)_*(y) \equiv \phi_*([y]) \pmod{J_n}$ , and
  - (b) for  $y \in J_n$ ,  $(i_n\phi)_*(y) = 0$ .
- (3) Let  $\varphi \in \mathcal{A}(BP)$  and suppose that  $\varphi_*(J_n) \subseteq J_n$ . Then, for  $z \in BP\langle n \rangle_*$ ,  $(p_n\varphi)_*(z) = [\varphi_*(z)]$ .

*Proof.* Part (1) is immediate from  $\pi_n\theta_n \simeq id$ . Then, for part (2), for  $y \in BP_*$ ,

$$(i_n\phi)_*(y) = (\theta_n)_*\phi_*(\pi_n)_*(y) = (\theta_n)_*\phi_*([y]) \equiv \phi_*([y]) \pmod{J_n},$$

and for  $y \in J_n$ ,

$$(i_n\phi)_*(y) = (\theta_n)_*\phi_*(\pi_n)_*(y) = (\theta_n)_*\phi_*(0) = 0.$$

Finally, for part (3), we have,

$$\begin{aligned} (p_n\varphi)_*(z) &= (\pi_n)_*\varphi_*(\theta_n)_*(z) \\ &= (\pi_n)_*\varphi_*(z + w) \quad \text{for some } w \in J_n \\ &= (\pi_n)_*(\varphi_*(z) + \varphi_*(w)) \quad \text{since } \varphi \text{ is additive} \\ &= [\varphi_*(z)] \quad \text{since, by hypothesis, } \varphi_*(w) \in J_n. \quad \square \end{aligned}$$

**Remark 2.5.** It is worth noting that  $(\theta_n)_*$  is *not* the obvious splitting on homotopy groups with image  $\mathbb{Z}_{(p)}[v_1, \dots, v_n]$  (and it is not a ring homomorphism). See [2, p817] for an example.

Another important consequence of the splitting is that the action of the additive  $BP\langle n \rangle$  operations of bidegree  $(0, 0)$  on homotopy groups is faithful. As we will see, the splitting allows us to deduce this from the corresponding result for  $BP$ , which was proved in [6, Proposition 1]. (Key ingredients for the  $BP$  case are that  $BP$ -theory has good duality and that everything is torsion-free.)

Given an unstable  $E$ -operation  $\theta \in E^0(\underline{E}_0) \cong [\underline{E}_0, \underline{E}_0]$ , we may consider the induced homomorphism of graded abelian groups  $\theta_* : \pi_*(\underline{E}_0) \rightarrow \pi_*(\underline{E}_0)$  given by the action of  $\theta$  on homotopy groups. For a graded abelian group  $M$ , we write  $\text{End}(M)$  for the ring of homomorphisms of graded abelian groups from  $M$  to itself.

Sending an operation to its action on homotopy groups gives a map

$$\begin{aligned} E^0(\underline{E}_0) &\rightarrow \text{End}(\pi_*(\underline{E}_0)) \\ \phi &\mapsto \phi_*. \end{aligned}$$

The restriction of this map to the additive  $E$ -operations  $\mathcal{A}(E)$  is a ring homomorphism and we denote this by  $\beta_E$ :

$$\begin{aligned} \beta_E : \mathcal{A}(E) &\rightarrow \text{End}(\pi_*(\underline{E}_0)) \\ \phi &\mapsto \phi_*. \end{aligned}$$

**Proposition 2.6.** *For all  $n \geq 0$ , the ring homomorphism*

$$\beta_{BP\langle n \rangle} : \mathcal{A}(BP\langle n \rangle) \rightarrow \text{End}(\pi_*(\underline{BP\langle n \rangle}_0))$$

*is injective.*

*Proof.* Let  $\phi \in \mathcal{A}(BP\langle n \rangle)$  and suppose that  $\beta_{BP\langle n \rangle}(\phi) = \phi_* = 0$ . Then

$$\beta_{BP}(i_n(\phi)) = (i_n(\phi))_* = (\theta_n \phi \pi_n)_* = (\theta_n)_* \phi_* (\pi_n)_* = 0.$$

But  $\beta_{BP}$  is injective (see [6, Proposition 1]) and so is  $i_n$ , so  $\phi = 0$ .  $\square$

### 3. THE COMPARISON MAP

In this section we begin with some reminders about the additive operations for the Adams summand  $g$  of  $p$ -local connective complex  $K$ -theory and we recall the main result of [6]. We then go on to define the main map to be studied in this paper,  $\hat{i}_n : \mathcal{A}(g) \rightarrow \mathcal{A}(BP\langle n \rangle)$ , and we discuss its basic properties.

A description of the ring of additive operations  $\mathcal{A}(g)$  for the Adams summand can be deduced from the corresponding result for integral complex  $K$ -theory (see [1, Lecture 4]). Another description can be found in [7]: Theorems 3.3 and 4.2 of [7] together give a topological basis for this ring, where the basis elements are certain polynomials in the Adams operations  $\Psi^0$ ,  $\Psi^p$  and  $\Psi^q$  (where  $q$  is primitive modulo  $p^2$  and thus a topological generator for the  $p$ -adic units). The precise details of the description are not needed here; what is important to note is that all operations can be described in terms of Adams operations.

The main result of [6] (in the split case) is the following.

**Theorem 3.1.** [6, Theorem 19] *There is an injective ring homomorphism  $\hat{i} : \mathcal{A}(g) \rightarrow \mathcal{A}(BP)$  such that the image is precisely the centre of the ring  $\mathcal{A}(BP)$ .*  $\square$

It is worth noting that  $\hat{i}$  is different from the ring homomorphism  $i_1 : \mathcal{A}(BP\langle 1 \rangle) = \mathcal{A}(g) \rightarrow \mathcal{A}(BP)$  provided by Lemma 2.2. Indeed,  $\hat{i}$  sends the identity operation of  $g$  to the identity operation of  $BP$ , whereas  $i_1$  does not. More generally, it is instructive to consider the effects of these two maps on Adams operations:  $\hat{i}$  takes the Adams operation  $\Psi_g^k$  of the Adams summand to the corresponding Adams operation  $\Psi_{BP}^k$  for  $BP$ ; this operation acts as multiplication by  $k^{(p-1)n}$  on each element of the group  $\pi_{2(p-1)n}(BP)$ . On the other hand,  $i_1$  sends  $\Psi_g^k$  to an operation which acts as zero on the ideal  $J_1$ .

The two maps share the property of being split by  $p_1$ .

**Lemma 3.2.** *The map  $p_1$  also splits  $\hat{i}$ .*

*Proof.* All elements of the topological ring  $\mathcal{A}(g)$  can be explicitly expressed as certain (infinite) linear combinations of Adams operations; see [6, Proposition 18]. By Lemma 2.2,  $p_1$  is additive, and it is straightforward to see that it is continuous with respect to the profinite filtrations on the rings of operations. Thus it is enough to check that  $p_1 \hat{i}(\Psi_g^k) = \Psi_g^k$  for all  $k \in \mathbb{Z}_{(p)}$ .

Now  $\hat{\iota}(\Psi_g^k) = \Psi_{BP}^k$  and this  $BP$  Adams operation acts as multiplication by  $k^{(p-1)r}$  on  $BP_{2(p-1)r}$  (and so, in particular, preserves  $J_1$ ). Then by part (3) of Lemma 2.4,  $p_1\hat{\iota}(\Psi_g^k)$  acts as multiplication by  $k^{(p-1)r}$  on  $\pi_{2(p-1)r}(g) = \mathbb{Z}_{(p)}\langle v_1^r \rangle$ . But, by [6, Proposition 1], this completely characterizes  $\Psi_g^k$ .  $\square$

The main map we will consider comes from composing the map  $\hat{\iota} : \mathcal{A}(g) \rightarrow \mathcal{A}(BP)$  of Theorem 3.1 with the map  $p_n : \mathcal{A}(BP) \rightarrow \mathcal{A}(BP\langle n \rangle)$  of Lemma 2.2.

**Definition 3.3.** Define  $\hat{\iota}_n = p_n\hat{\iota} : \mathcal{A}(g) \rightarrow \mathcal{A}(BP\langle n \rangle)$ .

Note that this gives us our map of operations without explicitly mentioning Adams operations for  $BP\langle n \rangle$ . On the other hand, we can define such Adams operations as follows.

**Definition 3.4.** Define unstable Adams operations for  $BP\langle n \rangle$  as the images of the corresponding  $BP$  operations:

$$\Psi_{BP\langle n \rangle}^k := p_n(\Psi_{BP}^k),$$

for  $k \in \mathbb{Z}_{(p)}$ .

Using part (3) of Lemma 2.4, we see that this definition gives unstable Adams operations for  $BP\langle n \rangle$  with the expected actions on homotopy (namely,  $\Psi_{BP\langle n \rangle}^k(z) = k^{(p-1)r}z$ , for  $z \in BP\langle n \rangle_{2(p-1)r}$ ).

Since  $\hat{\iota}(\Psi_g^k) = \Psi_{BP}^k$ , it follows from this definition of the Adams operations for  $BP\langle n \rangle$  and the description of  $\mathcal{A}(g)$  in terms of Adams operations, that the map  $\hat{\iota}_n$  is determined by mapping  $g$  Adams operations to the corresponding  $BP\langle n \rangle$  Adams operations and extending to (suitable infinite) linear combinations.

Our main result will be that the analogue of Theorem 3.1 holds for  $\hat{\iota}_n : \mathcal{A}(g) \rightarrow \mathcal{A}(BP\langle n \rangle)$ . We begin with some basic properties of  $\hat{\iota}_n$ ; in particular, it is a ring homomorphism (even though  $p_n$  is not).

**Proposition 3.5.** *For all  $n \geq 1$ , the map  $\hat{\iota}_n : \mathcal{A}(g) \rightarrow \mathcal{A}(BP\langle n \rangle)$  is an injective unital ring homomorphism whose image is contained in the centre of  $\mathcal{A}(BP\langle n \rangle)$ .*

*Proof.* First we check that  $\hat{i}_n$  is a ring homomorphism. For  $a, b \in \mathcal{A}(g)$ , we have

$$\begin{aligned}
i_n(\hat{i}_n(a)\hat{i}_n(b)) &= i_n(\hat{i}_n(a))i_n(\hat{i}_n(b)) && i_n \text{ ring homomorphism} \\
&= i_n p_n \hat{i}(a) i_n p_n \hat{i}(b) && \text{definition of } \hat{i}_n \\
&= e_n \hat{i}(a) e_n^2 \hat{i}(b) e_n && \text{by Lemma 2.2} \\
&= e_n^3 \hat{i}(a) \hat{i}(b) e_n && \text{image of } \hat{i} \text{ central} \\
&= e_n \hat{i}(a) \hat{i}(b) e_n && e_n \text{ idempotent} \\
&= e_n \hat{i}(ab) e_n && \hat{i} \text{ ring homomorphism} \\
&= i_n p_n \hat{i}(ab) && \text{by Lemma 2.2} \\
&= i_n \hat{i}_n(ab) && \text{definition of } \hat{i}_n.
\end{aligned}$$

But  $i_n$  is injective, so  $\hat{i}_n(a)\hat{i}_n(b) = \hat{i}_n(ab)$ .

Similarly, we find  $i_n(\hat{i}_n(1)) = e_n = i_n(1)$ , so  $\hat{i}_n(1) = 1$  and  $\hat{i}_n$  is unital.

Next we show injectivity. Let  $\phi \in \mathcal{A}(g)$ , with  $\phi \neq 0$ . By [6, Proposition 1], the action of operations in  $\mathcal{A}(g)$  on homotopy groups is faithful. Thus there is some  $r$  such that  $\phi$  acts on  $\pi_{2(p-1)r}(\underline{g}_0)$  as multiplication by some non-zero element  $\lambda$  of  $\mathbb{Z}_{(p)}$ . But then the action of  $\hat{i}_n(\phi)$  is given by multiplication by  $\lambda \neq 0$  on  $\pi_{2(p-1)r}(\underline{BP}\langle n \rangle_0) \neq 0$  and so  $\hat{i}_n(\phi) \neq 0$ .

Finally we need to see that the image is central. The image consists of certain infinite linear combinations of Adams operations for  $BP\langle n \rangle$ . It is clear from the action of  $\Psi_{BP\langle n \rangle}^k$  on homotopy that  $\beta_{BP\langle n \rangle}(\Psi_{BP\langle n \rangle}^k) = (\Psi_{BP\langle n \rangle}^k)_*$  commutes with all elements of  $\text{End}(\pi_*(\underline{BP}\langle n \rangle_0))$ . So the same holds for the image under  $\beta_{BP\langle n \rangle}$  of (suitable infinite) linear combinations of the Adams operations. But by Proposition 2.6,  $\beta_{BP\langle n \rangle}$  is injective, so any element of the image of  $\mathcal{A}(g)$  commutes with all elements of  $\mathcal{A}(BP\langle n \rangle)$ .  $\square$

As a consequence of the definitions, we have the following commutative diagram of abelian groups, for  $m \leq n$ , giving the compatibility between the various  $\hat{i}$  maps.

$$\begin{array}{ccccc}
& & \mathcal{A}(BP\langle n \rangle) & \xrightarrow[\cong]{i_n} & e_n \mathcal{A}(BP) e_n \\
& \nearrow \hat{i}_n & \uparrow p_n & & \downarrow e_m \circ - \circ e_m \\
\mathcal{A}(g) & \xrightarrow{\hat{i}} & \mathcal{A}(BP) & & \\
& \searrow \hat{i}_m & \downarrow p_m & & \\
& & \mathcal{A}(BP\langle m \rangle) & \xrightarrow[\cong]{i_m} & e_m \mathcal{A}(BP) e_m
\end{array}$$

**Remark 3.6.** It is natural to ask if one can obtain the ring  $\mathcal{A}(BP)$  as any kind of limit over the  $\mathcal{A}(BP\langle n \rangle)$ , but this does not seem to be the case. On the one hand, we can put the  $\mathcal{A}(BP\langle n \rangle)$  into a direct system of injective ring homomorphisms and produce an injective ring homomorphism

$\lim_{\rightarrow_n} \mathcal{A}(BP\langle n \rangle) \rightarrow \mathcal{A}(BP)$ . However, this is not surjective; for example the identity operation on  $BP$  is not in the image. On the other hand, the maps in the other direction are not ring homomorphisms, so the inverse limit  $\lim_{\leftarrow_n} \mathcal{A}(BP\langle n \rangle)$  can only be formed in the category of abelian groups.

#### 4. DIAGONAL OPERATIONS

We define unstable diagonal operations for  $BP\langle n \rangle$ , just as was done for  $BP$  in [6].

**Definition 4.1.** Write  $\mathcal{D}(BP\langle n \rangle)$  for the subring of  $\mathcal{A}(BP\langle n \rangle)$  consisting of operations whose action on each homotopy group  $\pi_{2(p-1)r}(BP\langle n \rangle_0)$  is multiplication by an element  $\mu_r$  of  $\mathbb{Z}_{(p)}$ . We call elements of  $\mathcal{D}(BP\langle n \rangle)$  *unstable diagonal operations*.

The main result of this section will be that the central operations coincide with the diagonal operations. One inclusion is easy.

For a ring  $R$ , we write  $Z(R)$  for its centre.

**Lemma 4.2.** *We have  $\mathcal{D}(BP\langle n \rangle) \subseteq Z(\mathcal{A}(BP\langle n \rangle))$ .*

*Proof.* The action on homotopy of  $\phi \in \mathcal{D}$  commutes with the action of any operation in  $\mathcal{A}(BP\langle n \rangle)$ , so the inclusion  $\mathcal{D}(BP\langle n \rangle) \subseteq Z(\mathcal{A}(BP\langle n \rangle))$  follows from the faithfulness of the action (Proposition 2.6).  $\square$

Our proof of the reverse inclusion will amount to finding enough operations in order to force a central operation to act diagonally. Our strategy will be to start from stable  $BP$  operations, over which we have better control, and then to view these as additive unstable operations and project them to  $\mathcal{A}(BP\langle n \rangle)$ .

First we will need some notation for sequences indexing monomials. We write  $v^\alpha$  for the monomial  $v_1^{\alpha_1} v_2^{\alpha_2} \dots v_m^{\alpha_m}$ , where  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$  is a sequence of non-negative integers, with  $\alpha_m \neq 0$ . We order such sequences right lexicographically; explicitly for  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$  and  $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ , we have  $\alpha < \beta$  if  $m < n$  or if  $m = n$  and there is some  $j$ , with  $1 \leq j \leq m$ , such that  $\alpha_k = \beta_k$  for all  $k > j$  but  $\alpha_j < \beta_j$ .

We add sequences placewise:  $(\alpha + \beta)_i = \alpha_i + \beta_i$ , so that  $v^\alpha v^\beta = v^{\alpha + \beta}$ . It is straightforward to check that the ordering behaves well with respect to the addition: if  $\alpha \leq \alpha'$  and  $\beta \leq \beta'$  then  $\alpha + \beta \leq \alpha' + \beta'$ .

The degree of  $v^\alpha$  is  $2 \sum_{i=1}^m \alpha_i (p^i - 1)$  and we write this as  $|\alpha|$ .

**Lemma 4.3.** *Let  $\alpha, \beta, \gamma$  denote sequences indexing monomials in the same degree,  $|\alpha| = |\beta| = |\gamma|$ .*

- (1) *There is a stable  $BP$  operation  $\phi_\beta$  in  $BP^{|\alpha|}(BP)$  whose action  $BP_{|\alpha|} \rightarrow BP_0 = \mathbb{Z}_{(p)}$  has the property that  $(\phi_\beta)_*(v^\gamma) = \mu_{\gamma, \beta}$ , where*

$$\begin{aligned} \mu_{\beta, \beta} &\neq 0, \\ \mu_{\gamma, \beta} &= 0 \quad \text{if } \gamma < \beta. \end{aligned}$$



- (2) *There is a stable BP operation  $\phi_{\alpha,\beta}$  in  $BP^0(BP)$  whose action  $BP_{|\alpha|} \rightarrow BP_{|\alpha|}$  has the property that  $(\phi_{\alpha,\beta})_*(v^\gamma) = \mu_{\gamma,\beta}v^\alpha$ , where*

$$\begin{aligned}\mu_{\beta,\beta} &\neq 0, \\ \mu_{\gamma,\beta} &= 0 \quad \text{if } \gamma < \beta.\end{aligned}$$

*Proof.* The second part follows immediately from the first, by taking  $\phi_{\alpha,\beta} = v^\alpha \phi_\beta$ .

For the first part, we recall that  $BP$  has good duality and so a stable operation  $\phi$  in  $BP^*(BP)$  corresponds to a degree zero  $BP_*$ -linear functional  $\bar{\phi} : BP_*(BP) \rightarrow BP_*$ . The action of the operation on coefficient groups is recovered from the functional by precomposition with the right unit map,  $\eta_R : BP_* \rightarrow BP_*(BP)$ ; that is,  $\phi_* = \bar{\phi}\eta_R$ . We have  $BP_*(BP) = BP_*[t_1, t_2, \dots]$  and so a functional as described above is determined by any choice of its value on each monomial in the  $ts$ .

The map  $\eta_R$  is of course very complicated, but we will only need to exploit some basic information about its form. We have

$$\eta_R(v_m) = pt_m + \sum \lambda_\gamma t^\gamma + \sum \mu_{\delta,\delta'} v^\delta t^{\delta'},$$

where  $\lambda_\gamma, \mu_{\delta,\delta'} \in \mathbb{Z}_{(p)}$ ,  $\delta \neq \emptyset$ , and  $\gamma$  runs over sequences other than  $(0, \dots, 0, 1)$  in the degree of  $v_m$ . (The only content here is the form of the top term, of course.) Now  $\eta_R$  is a ring map and it follows from the properties of the ordering on monomials described above that

$$\eta_R(v^\gamma) = \lambda t^\gamma + \sum_{\gamma' < \gamma} \lambda'_{\gamma'} t^{\gamma'} + \sum_{\delta \neq \emptyset} \mu'_{\delta,\delta'} v^\delta t^{\delta'},$$

for some  $\lambda, \lambda'_{\gamma'}, \mu'_{\delta,\delta'} \in \mathbb{Z}_{(p)}$  with  $\lambda \neq 0$ .

Now consider the functional  $\bar{\phi}_\beta : BP_*(BP) \rightarrow BP_*$  which is zero on all monomials except  $t^\beta$  and sends  $t^\beta$  to 1. By construction the corresponding operation  $\phi_\beta$  has the required property.  $\square$

Now the following lemma follows as a matter of elementary linear algebra. Let  $E_{\alpha,\beta}$  denote the elementary matrix with a 1 in the  $(\alpha, \beta)$  position and zeroes everywhere else.

**Lemma 4.4.** *For all  $\alpha, \beta$  with  $|\alpha| = |\beta|$ , there is some non-zero  $\bar{\mu}_{\alpha,\beta} \in \mathbb{Z}_{(p)}$  and an operation  $\varphi_{\alpha,\beta}$  in  $BP^0(BP)$  such that the matrix of its action on  $BP_{|\alpha|}$  is  $\bar{\mu}_{\alpha,\beta} E_{\alpha,\beta}$ .*

*Proof.* The preceding lemma gives the operation  $\phi_{\alpha,\beta}$ . Using the  $\mathbb{Z}_{(p)}$ -basis of monomials in the  $vs$ , ordered as above, this operation acts on coefficients in the given degree by the matrix

$$M_{\alpha,\beta} = \sum_{\gamma \geq \beta} \mu_{\gamma,\beta} E_{\alpha,\gamma},$$

where  $\mu_{\beta,\beta} \neq 0$ .

If we order the elementary matrices by  $E_{\beta,\gamma} < E_{\beta',\gamma'}$  if  $\gamma < \gamma'$  or  $\gamma = \gamma'$  and  $\beta < \beta'$ , then the above shows that the matrix writing the  $M_{\alpha,\beta}$  in terms

of the  $E_{\alpha,\beta}$  is non-singular lower triangular. Hence, for some  $\bar{\mu}_{\alpha,\beta} \neq 0$ , we can write  $\bar{\mu}_{\alpha,\beta}E_{\alpha,\beta}$  as a  $\mathbb{Z}_{(p)}$ -linear combination of the  $M_{\alpha,\beta}$ . We take  $\varphi_{\alpha,\beta}$  to be the corresponding linear combination of the  $\phi_{\alpha,\beta}$ .  $\square$

**Theorem 4.5.** *We have  $Z(\mathcal{A}(BP\langle n \rangle)) = \mathcal{D}(BP\langle n \rangle)$ .*

*Proof.* We noted the inclusion  $\mathcal{D}(BP\langle n \rangle) \subseteq Z(\mathcal{A}(BP\langle n \rangle))$  in Lemma 4.2 above, so it remains to show the reverse inclusion.

As in the proof of Lemma 11 of [6], there is an injection  $BP^0(BP) \hookrightarrow \mathcal{A}(BP)$  from the stable degree zero  $BP$  operations to the additive unstable bidegree  $(0,0)$  operations, given by sending a stable operation to its zero component (that is, applying  $\Omega^\infty$ ). This allows us to view the operation  $\varphi_{\alpha,\beta}$  constructed above as an element of  $\mathcal{A}(BP)$ , still acting on coefficients in the specified degree as some non-zero multiple of the elementary matrix  $E_{\alpha,\beta}$ .

Next we map these operations to  $\mathcal{A}(BP\langle n \rangle)$ : consider  $p_n(\varphi_{\alpha,\beta}) \in \mathcal{A}(BP\langle n \rangle)$ . We consider the action of this operation on coefficients in degree  $|\alpha|$ . Now we can write  $BP_{|\alpha|}$  as a direct sum of  $\mathbb{Z}_{(p)}$ -modules  $R \oplus J$ , where  $R = BP_{|\alpha|} \cap \mathbb{Z}_{(p)}[v_1, \dots, v_n]$  and  $J = BP_{|\alpha|} \cap J_n$ . Notice that any monomial in the  $vs$  lying in  $R$  is lower in the ordering than any monomial lying in  $J$ . So, when we write the action of an operation as a matrix with respect to the monomial basis, this splits into blocks, according to the decomposition into  $R$  and  $J$ .

Now let  $\alpha, \beta$  index monomials in  $R$ . Using  $(p_n(\varphi_{\alpha,\beta}))_* = (\pi_n)_*(\varphi_{\alpha,\beta})_*(\theta_n)_*$ , it is easy to check the action of  $p_n(\varphi_{\alpha,\beta})$  is given by  $\bar{\mu}_{\alpha,\beta}E_{\alpha,\beta}$  on  $BP\langle n \rangle_{|\alpha|}$ .

So now suppose we have a central operation  $\phi \in \mathcal{A}(BP\langle n \rangle)$ . Since it commutes with each operation  $\varphi_{\alpha,\beta}$ , its action on  $BP\langle n \rangle_{|\alpha|}$  commutes with the action of some non-zero multiple of each elementary matrix. Hence the matrix of its action in this degree is diagonal with all diagonal entries equal. That is  $\phi \in \mathcal{D}(BP\langle n \rangle)$ .  $\square$

## 5. CONGRUENCES

Let  $S_g$  be the subring of the infinite direct product  $\prod_{i=0}^\infty \mathbb{Z}_{(p)}$  consisting of sequences  $(\mu_i)_{i \geq 0}$  satisfying the system of congruences which characterizes the action on coefficient groups of an element of  $\mathcal{A}(g)$ .

The congruences can be described as follows; for further details see [6, Section 4]. Let  $G$  denote the periodic Adams summand and let  $\{\hat{f}_n \mid n \geq 0\}$  be a  $\mathbb{Z}_{(p)}$ -basis for  $QG_0(\underline{G}_0)$ , where  $Q$  denotes the indecomposable quotient for the  $\star$ -product. These basis elements can be written as rational polynomials in the variable  $\hat{w} = \hat{u}^{-1}\hat{v}$ , where  $G_* = \mathbb{Z}_{(p)}[\hat{u}^{\pm 1}]$  and  $\hat{v} = \eta_R(\hat{u})$ . The  $n$ -th congruence is the condition that the rational linear combination of the  $\mu_i$  obtained from  $\hat{f}_n$  by sending  $\hat{w}^i$  to  $\mu_i$  lies in  $\mathbb{Z}_{(p)}$ . Different choices of basis lead to equivalent systems of congruences with the same solution set  $S_g$ . (Explicit choices, involving Stirling numbers, are known, but we do not need these here.)

The following proposition is a stronger version of the congruence result of [6]. The proof closely follows that of [6, Proposition 16].

**Proposition 5.1.** *Fix  $n \geq 1$ . Suppose that an operation  $\theta \in \mathcal{A}(BP)$  is such that its action on homotopy  $\theta_* : BP_* \rightarrow BP_*$  satisfies the following conditions. For each  $i \geq 0$ , there is some  $\mu_i \in \mathbb{Z}_{(p)}$  such that*

- (1)  $\theta_*(x) \equiv \mu_i x \pmod{J_n}$  if  $x \notin J_n$ ,  $|x| = 2(p-1)i$ , and
- (2)  $\theta_*(x) = 0$  if  $x \in J_n$ .

Then  $(\mu_i)_{i \geq 0} \in S_g$ .

*Proof.* Under the isomorphism  $PBP^0(\underline{BP}_0) \cong \text{Hom}_{BP_*}(QBP_*(\underline{BP}_0), BP_*)$ , the operation  $\theta$  corresponds to a  $BP_*$ -linear functional  $\bar{\theta} : QBP_*(\underline{BP}_0) \rightarrow BP_*$  of degree zero.

Let  $V_\mu : QBP_*(\underline{BP}_0) \rightarrow \mathbb{Z}_{(p)}$  be the composite  $\pi \bar{\theta}$  where  $\pi : BP_* \rightarrow \mathbb{Z}_{(p)}$  is defined to be the ring map determined by

$$\begin{aligned} v_1 &\mapsto 1, \\ v_i &\mapsto 0, \quad \text{for } i > 1. \end{aligned}$$

Thus we have a commutative diagram

$$\begin{array}{ccc} QBP_*(\underline{BP}_0) & \xrightarrow{\bar{\theta}} & BP_* \\ & \searrow V_\mu & \downarrow \pi \\ & & \mathbb{Z}_{(p)} \end{array}$$

Recall from [2] that  $QBP_*(\underline{BP}_0)$  is torsion-free and rationally generated by elements of the form  $v^\alpha e^{2(p-1)h} \eta_R(v^\beta)$ , where  $v^\alpha \in BP_*$ ,  $v^\beta \in BP_{2(p-1)h}$ ,  $e \in QBP_1(\underline{BP}_1)$  is the suspension element and  $\eta_R$  is the right unit map. By [2, 12.4], the action of an operation  $\theta$  on homotopy can be recovered from the corresponding functional  $\bar{\theta}$  via  $\theta_*(v^\beta) = \bar{\theta}(e^{2(p-1)h} \eta_R(v^\beta))$ , for  $v^\beta \in BP_{2(p-1)h}$ .

We have

$$\begin{aligned} V_\mu(v^\alpha e^{2(p-1)h} \eta_R(v^\beta)) &= \pi \bar{\theta}(v^\alpha e^{2(p-1)h} \eta_R(v^\beta)) \\ &= \pi(v^\alpha \theta_*(v^\beta)) \\ &= \begin{cases} \pi(v^\alpha (\mu_h v^\beta + y)) & \text{for some } y \in J_n, \text{ if } v^\beta \notin J_n \\ 0 & \text{if } v^\beta \in J_n \end{cases} \\ &= \begin{cases} \mu_h & \text{if } \alpha = (\alpha_1, 0, 0, \dots) \text{ and } \beta = (h, 0, 0, \dots) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus for each  $x \in QBP_*(\underline{BP}_0)$ ,  $V_\mu(x)$  is some rational linear combination of the  $\mu_i$  and since this lies in  $\mathbb{Z}_{(p)}$ , this gives congruences which must be satisfied by the  $\mu_i$ .

Consider the standard map of ring spectra  $BP \rightarrow G$ . This induces a map of Hopf rings  $BP_*(BP_*) \rightarrow G_*(G_*)$  and thus a ring map on indecomposables  $QBP_*(BP_0) \rightarrow QG_*(G_0)$  which we denote by  $\phi$ .

Now we claim that we can factorize  $V_\mu$  as  $\pi_\mu \tilde{\phi}$  where  $\tilde{\phi} : QBP_*(BP_0) \rightarrow \text{Im}(\phi)$  is the map given by restricting the codomain of  $\phi : QBP_*(BP_0) \rightarrow QG_*(G_0)$ , and

$$\pi_\mu : \text{Im}(\phi) \rightarrow \mathbb{Z}_{(p)}$$

is the  $\mathbb{Q}$ -linear map determined by

$$\hat{u}^a e^{2(p-1)b} \hat{v}^b \mapsto \mu_b.$$

To prove the claim, it is enough to check on rational generators:

$$\begin{aligned} & \pi_\mu \tilde{\phi}(v^\alpha e^{2(p-1)h} \eta_R(v^\beta)) \\ &= \begin{cases} \pi_\mu(\hat{u}^{\alpha_1} e^{2(p-1)h} \hat{v}^h) & \text{if } \alpha = (\alpha_1, 0, 0, \dots) \text{ and } \beta = (h, 0, 0, \dots) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \mu_h & \text{if } \alpha = (\alpha_1, 0, 0, \dots) \text{ and } \beta = (h, 0, 0, \dots) \\ 0 & \text{otherwise} \end{cases} \\ &= V_\mu(v^\alpha e^{2(p-1)h} \eta_R(v^\beta)). \end{aligned}$$

Just as in [6, proof of Theorem 19], up to some shift by a power of  $\hat{u}$ , each basis element of  $QG_0(G_0)$ , say  $\hat{f}_n$ , for  $n \geq 0$ , is in the image of the map from  $QBP_*(BP_0)$ . So we have  $x_n \in QBP_*(BP_0)$  such that  $\phi(x_n) = \hat{u}^{c_n} \hat{f}_n$ , for some  $c_n \in \mathbb{Z}$ . Then  $V_\mu(x_n) = \pi_\mu \tilde{\phi}(x_n) = \pi_\mu(\hat{u}^{c_n} \hat{f}_n)$ .

But  $V_\mu(x_n) \in \mathbb{Z}_{(p)}$  and  $\pi_\mu(\hat{u}^{c_n} \hat{f}_n) \in \mathbb{Z}_{(p)}$  is exactly the  $n$ -th congruence condition for  $g$ . Hence  $(\mu_i)_{i \geq 0} \in S_g$ .  $\square$

Now we show how this applies to  $BP\langle n \rangle$  operations.

**Proposition 5.2.** *Let  $n \geq 1$  and  $\phi \in \mathcal{D}(BP\langle n \rangle)$ . Then  $i_n(\phi) \in \mathcal{A}(BP)$  satisfies the hypotheses of Proposition 5.1.*

*Proof.* Let  $\phi \in \mathcal{D}(BP\langle n \rangle)$ , where the action of  $\phi_*$  on  $\pi_{2(p-1)i}(BP\langle n \rangle_0)$  is multiplication by  $\mu_i$ . Then, using  $[-]$  to denote classes modulo  $J_n$ , for  $v^\alpha \in BP_*$ , by part (2) of Lemma 2.4,

$$\begin{aligned} (i_n(\phi))_*(v^\alpha) &= \begin{cases} \phi_*([v^\alpha]) \mod J_n & \text{if } v^\alpha \notin J_n \\ 0 & \text{if } v^\alpha \in J_n \end{cases} \\ &= \begin{cases} \mu_{\|\alpha\|} v^\alpha \mod J_n & \text{if } v^\alpha \notin J_n \\ 0 & \text{if } v^\alpha \in J_n, \end{cases} \end{aligned}$$

where  $\|\alpha\| = \frac{|\alpha|}{2(p-1)}$ .  $\square$

Putting everything together gives the following.

**Theorem 5.3.** *For all  $n \geq 1$ , the image of the injective ring homomorphism  $\hat{i}_n : \mathcal{A}(g) \hookrightarrow \mathcal{A}(BP\langle n \rangle)$  is the centre  $Z(\mathcal{A}(BP\langle n \rangle))$  of  $\mathcal{A}(BP\langle n \rangle)$ .*

*Proof.* We have  $\text{Im}(\hat{i}_n) \subseteq Z(\mathcal{A}(BP\langle n \rangle)) = \mathcal{D}(BP\langle n \rangle)$ , by Proposition 3.5 and Theorem 4.5. Now let  $\phi \in \mathcal{D}(BP\langle n \rangle)$ , where  $\phi$  acts on  $\pi_{2(p-1)i}(\overline{BP\langle n \rangle}_0)$  as multiplication by  $\mu_i$ . By Proposition 2.6,  $\phi$  is completely determined by the sequence  $(\mu_i)_{i \geq 0}$ . By Proposition 5.2,  $i_n(\phi)$  satisfies the hypotheses of Proposition 5.1 for the same sequence  $(\mu_i)_{i \geq 0}$ , and so by Proposition 5.1,  $(\mu_i)_{i \geq 0} \in S_g$ . Thus we have the following commutative diagram.

$$\begin{array}{ccccc}
 \mathcal{A}(g) & \xrightarrow[\hat{i}_n]{\cong} & \text{Im}(\hat{i}_n) & \hookrightarrow & Z(\mathcal{A}(BP\langle n \rangle)) & \xrightarrow{=} & \mathcal{D}(BP\langle n \rangle) \\
 \cong \downarrow & & & & & & \downarrow \\
 S_g & & & \xrightarrow{=} & & & S_g
 \end{array}$$

So the inclusions are equalities. □

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